Section 14.5

Curvature

We have looked at the derivative $\vec{v}(t) = \vec{r}\,'(t)$, giving the rate at which a particle moves along the curve $\vec{r}$ at time t.

Rather than just looking at movement of a particle, we can also look at how the curve itself changes direction.

We need a tangent vector for $\vec{r}(t)$ that does not depend on the time it takes to traverse a given segment of $\vec{r}$

Definition:
The rate of change of the tangent direction of a smooth curve $\vec{r}(s)$ is $\dfrac{d\vec{T}}{ds}$. The **curvature** of $\vec{r}$ is $\kappa(s) = \left|\dfrac{d\vec{T}}{ds}\right|$.

Example 14.5.1. Find the curvature of $\vec{r}(t) = \langle R\cos(t/R), R\sin(t/R)\rangle$

What if we cannot reparameterize $\vec{r}$ in terms of its arc length?

The speed of change in the direction of $\vec{r}$ is given by κ, which measures the amount of turning in the curve. The direction of the turning is measured by the unit vector in the direction of $\dfrac{d\vec{T}}{ds}$.

Definition: Principal Unit Normal Vector
Let $\vec{r}$ describe a smooth curve parameterized by arc length. The **principal unit normal vector** at a point P on the curve at which $\kappa \neq 0$ is

$$N(s) = \frac{d\vec{T}/ds}{|d\vec{T}/ds|} = \frac{1}{\kappa}\frac{d\vec{T}}{ds}$$

If $t = f(s)$, with $\vec{r}(t) = P$, the principal unit normal vector at P is

$$N(t) = \frac{d\vec{T}/dt}{|d\vec{T}/dt|}$$

Example 14.5.2. Find the unit tangent and principal unit normal vectors to the helix of $a, b > 0$.

$$\vec{r}(t) = \langle -a\sin t, a\cos t, b\rangle$$

Relating back to particle motion along $\vec{r}(t)$, the acceleration $\vec{a}(t) = \vec{r}\,''(t)$ is given by a linear combination of N and T.

The unit tangent vector $\vec{T}$, the principal unit normal vector $\vec{N}$ and the acceleration $\vec{a}$ lie in a common plane called the **osculating plane**. We wish to measure how quickly the curve $\vec{r}$ leaves this plane, or equivalently, how quickly said plane changes with $\vec{T}(s)$ and $\vec{N}(s)$.

Definition:
Let C be a smooth parameterized curve with unit tangent and principal unit normal vectors $\vec{T}$ and $\vec{N}$. Then at each point of the curve at which the $\kappa \neq 0$, the **unit binormal vector** is

$$\vec{B} = \vec{T} \times \vec{N}.$$

The **torsion** is

$$\tau = -\frac{d\vec{B}}{ds} \cdot \vec{N}.$$